MULTIVARIATE ANALYSIS OF VARIANCE FOR A SPECIAL COVARIANCE CASE

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Multivariate analysis of variance tests are developed for situations where the underlying covariance structure is uniform (equal variances and covariances) in terms of statistics analogous to Hotelling's T^2 and T_0^2. Extensions are made to several populations as well as to blocks of uniform covariance matrices. A special case, which is typical of the test procedures given here, is the problem of testing whether the mean vector of a bivariate normal distribution is equal to some specified vector based on n observations. The uniform structure assumes that the two unknown variances are equal though the correlation is arbitrary. The testing procedure leads to a statistic U which is distributed as the sum of two independent $F_{1,n-1}$ ratios which may be contrasted with the T^2 statistic proportional to $F_{2,n-2}$ used in the situation where the variances are not assumed equal.

INTRODUCTION

In general a $k\times k$ covariance matrix of a multivariate normal distribution is specified by $k(k+1)/2$ different parameters. In certain instances the number of different parameters can be considerably reduced. We will consider here multivariate hypothesis testing for a special reduced parameter covariance situation that we shall call the uniform case of order m, i.e., $(u)_m$. Certain applications of this case to a general serial correlation model will be given when $k=3, 4, 5$ for testing the null hypothesis as in the analysis of variance that the k means are all equal. This is in a sense analogous to the mixed model analysis of variance situation, where the errors are not independent but are serially related. Previous work in the area of special covariance matrices by Wilks [9] and Votaw [8] was focused on problems of symmetry and compound symmetry, with test statistics derived on the basis of the likelihood ratio criterion. Here, the main interest is in testing whether a vector mean is equal to some specified value, or whether g vector means are equal given a uniform covariance structure.

1. THE UNIFORM CASE $(u)_1$

Let $x'=(x_1, x_2, \cdots, x_k)$ have a k-variate normal distribution with vector mean $\mu'=(\mu_1, \cdots, \mu_k)$ and uniform covariance matrix

$$\Lambda = \sigma^2 \begin{pmatrix} 1 & \rho & \cdot & \cdot & \cdot & \cdot & \rho \\ \rho & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \rho \\ \rho & \cdot & \cdot & \cdot & \cdot & \rho & 1 \end{pmatrix}. \qquad (1.1)$$

Let a sample of n observations be drawn at random from this population. The hypothesis we wish to test is

$$H_0: \mu' = (\mu_{10}, \cdots, \mu_{k0})$$
$$H_1: \mu' \neq \mu_0'.$$

Let $\bar{x}' = (\bar{x}_1, \cdots, \bar{x}_k)$, be the sample mean vector and

$$S_u = \begin{bmatrix} w & z & \cdot & \cdot & \cdot & z \\ z & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & z \\ z & \cdot & \cdot & \cdot & z & w \end{bmatrix} \tag{1.2}$$

where w is the average of the sample variances and z is the average of the sample covariances.

Now a monotonic function of the likelihood ratio test statistic is

$$\lambda = \left| I + S_u^{-1} B \right|^{-1} \tag{1.3}$$

where

$$B = \begin{bmatrix} a & b & \cdot & \cdot & \cdot & b \\ b & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & b \\ b & \cdot & \cdot & \cdot & b & a \end{bmatrix}$$

$$a = n(n-1)^{-1}(k-1)^{-1}(\bar{x} - \mu_0)'(\bar{x} - \mu_0),$$
$$b = n(n-1)k^{-1}(k-1)^{-1}(\bar{x} - \mu_0)'(E - I)(\bar{x} - \mu_0),$$

and E is a $k \times k$ matrix all of whose elements are unity. Since the characteristic roots of $S_u^{-1}B$ are $(a-b)/(w-z)$ which is of multiplicity $k-1$ and $[a+(k-1)b]/[w+(k-1)z]$,

$$\lambda = \left(1 + \frac{a-b}{w-z}\right)^{-(k-1)} \left(1 + \frac{a+(k-1)b}{w+(k-1)z}\right)^{-1}. \tag{1.4}$$

In the null case $a-b$, $a+(k-1)b$, $w-z$, and $w+(k-1)z$ are mutually independent and distributed like $(n-1)^{-1}(k-1)^{-1}\chi^2_{k-1}$, $(n-1)^{-1}\chi^2_1$, $(n-1)^{-1}(k-1)^{-1}\chi^2_{(k-1)(n-1)}$, and $(n-1)^{-1}\chi^2_{n-1}$, respectively. It then follows that

$$\lambda = (1+(n-1)^{-1}F_{k-1,(n-1)(k-1)}(1 + (n-1)^{-1}F_{1,n-1})^{-1}, \tag{1.5}$$

or that λ is distributed like the product of two independent variables $u^{k-1}v$

where

$$f(u) \propto (1-u)^{(k-3)/2} u^{((n-1)(k-1)-2)/2}$$
$$f(v) \propto (1-v)^{-1/2} v^{(n-3)/2}. \tag{1.6}$$

The moments of λ can be easily computed so that

$$E\lambda^r = \frac{\Gamma\left(\frac{n(k-1)}{2}\right)\Gamma\left(\frac{(k-1)(n-1+2r)}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1+2r}{2}\right)}{\Gamma\left(\frac{(k-1)(n-1)}{2}\right)\Gamma\left(\frac{(k-1)(n+2r)}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+2r}{2}\right)}. \tag{1.7}$$

Useful approximations to the product of beta variables are given by Tukey and Wilks [7].

A second criterion which suggests itself is to test the F ratios separately and reject the null hypothesis if at least one of the F ratios is significant.

Another alternative statistic is the analogue of Hotelling's T^2 or its generalization T_0^2 [4] which is based on the trace of the product of two matrices. Actually this statistic may be derived from the information criterion of Kullback [6, chap. 9]. This criterion yields here,

$$U = ntrS_u^{-1}(\bar{x}-\mu_0)(\bar{x}-\mu_0)' = n(\bar{x}-\mu_0)'S_u^{-1}(\bar{x}-\mu_0), \tag{1.8}$$

which can be written as

$$U = \frac{(n-1)(k-1)(a-b)}{w-z} + \frac{(n-1)(a+(k-1)b)}{w+(k-1)z}, \tag{1.9}$$

and hence results in

$$U = (k-1)F_{k-1,(k-1)(n-1)} + F_{1,n-1}, \tag{1.10}$$

where the F ratios are independent. Therefore in the null case U, the natural analogue of T^2, is distributed as a linear sum of independent F ratios and tends asymptotically to χ_k^2 as $n \to \infty$. In the alternative case at least one of the F ratios is a non-central F.

From this point on we shall restrict our attention to the statistics which can be derived by use of the information criterion. The computation of the information statistic U can be made by resorting to an analysis of variance table. Let $x_{ij} - \mu_{j0} = y_{ij}$, $i = 1, \cdots, n$, $j = 1, \cdots, k$, where the variables are the columns and the vector observations the rows.

From Table 1 we get

$$U = (n-1)\left[\frac{(k-1)Q_1}{Q_3} + \frac{Q_4}{Q_2}\right]. \tag{1.11}$$

A confidence ellipsoid for the mean vector μ is obtained in the same way as for the T^2 statistic (Anderson [2, p. 108]), only here one would use the percentage points of U.

TABLE 1

Source	D.F.	Sum of Squares	M.S.
Columns	$k-1$	$Q_1=n\sum_{j=1}^{k}(\bar{y}_{.j}-\bar{y}_{..})^2$	$(k-1)^{-1}Q_1$
Rows	$n-1$	$Q_2=k\sum_{i=1}^{n}(\bar{x}_{i.}-\bar{x}_{..})^2$	$(n-1)^{-1}Q_2$
C×R	$(k-1)(n-1)$	$Q_3=\sum_j\sum_i(x_{ij}-\bar{x}_{.j}-\bar{x}_{i.}+x_{..})^2$	$(k-1)^{-1}(n-1)^{-1}Q_3$
Grand Mean	1	$Q_4=n\,k\,\bar{y}^2_{..}$	Q_4

We may extend this to g populations, i.e., multivariate analysis of variance. We take a random sample of n_α observations from the αth population $\alpha=1, \cdots g$, where we assume that the g k-variate normal populations have the common uniform covariance matrix of (1.1). Let $\eta'_\alpha=(\eta_{\alpha 1}, \cdots, \eta_{\alpha k})$ be the vector mean of the αth population and we wish to test whether the $\eta_1=\cdots=\eta_k$. In this case the information statistic turns out to be distributed as a linear sum of two independent F ratios under the null hypothesis, so that

$$U_g=(g-1)(k-1)F_{(k-1)(g-1),(k-1)(N-g)}+(g-1)F_{g-1,N-g}, \qquad (1.12)$$

where $N=\sum n_\alpha$ and $g\geq 2$. U_g is asymtotically $\chi^2_{k(g-1)}$. Under the alternative hypothesis, at least one of the F's is a noncentral F.

The computation of U can also be obtained from an analysis of variance table where $x_{ij\alpha}$ are the observations $i=1, \cdots, n_\alpha; j=1, \cdots, k; \alpha=1, \cdots, g$. where the variables are the columns and the rows are the vector observations,

TABLE 2

Source	D.F.	Sum of Squares	M.S.
Col.	$k-1$	$Q_1=N\sum_{j=1}^{k}(\bar{x}_{.j.}-\bar{x}_{...})^2$	$(k-1)^{-1}Q_1$
Populations	$g-1$	$Q_2=k\sum_\alpha n_\alpha(\bar{x}_{..\alpha}-\bar{x}_{...})^2$	$(g-1)^{-1}Q_2$
Rows within Pop.	$N-g$	$Q_3=k\sum_\alpha\sum_i(\bar{x}_{i.\alpha}-\bar{x}_{..\alpha})^2$	$(N-g)^{-1}Q_3$
Col.×Pop.	$(k-1)(g-1)$	$Q_4=\sum_\alpha n_\alpha\sum_j(\bar{x}_{.j\alpha}-\bar{x}_{..\alpha}-\bar{x}_{.j.}+\bar{x}_{...})^2$	$(k-1)^{-1}(g-1)^{-1}Q_4$
Col. × rows within Pop.	$(k-1)(N-g)$	$Q_5=\sum_\alpha\sum_j\sum_i(x_{ij\alpha}-\bar{x}_{.j\alpha}-\bar{x}_{i.\alpha}+\bar{x}_{..\alpha})^2$	$(k-1)^{-1}(N-g)^{-1}Q_5$
Total	$Nk-1$		

From Table 2 we obtain

$$U_g=(N-g)\left[\frac{(k-1)Q_4}{Q_5}+\frac{Q_2}{Q_3}\right]. \qquad (1.13)$$

2. THE UNIFORM CASE OF ORDER 2 $(\mu)_2$

Let $x' = (x_1', x_2')$ be a multivariate normal distribution with mean $\mu = (\mu_1', \mu_2')$ and partitioned covariance matrix

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{pmatrix} \tag{2.1}$$

where

$$x_i' = (x_{1i}, \cdots, x_{k_i i}),\ \mu_i' = (\mu_{1i}, \cdots, \mu_{k_i i}),\ \Lambda_{ii} = (\Phi_i - \beta_i)I_i + \beta_i E_i \quad \text{for } i = 1, 2;\ \Lambda_{12} = \beta_{12}E_{12},$$

where I_i is the $k_i \times k_i$ identity matrix, E_i is the $k_i \times k_i$ matrix all of whose elements are unity and E_{12} is the $k_1 \times k_2$ matrix all of whose elements are unity. We shall call this covariance structure uniform of order 2 or $(u)_2$, since it is essentially made up of two partitioned uniform covariance matrices.

A sample of n observations from this population provides a test of the hypothesis that $\mu' = \mu_0 = (\mu_{10}, \mu_{20})$. Letting $\bar{x}_i' = (\bar{x}_{1i}, \cdots, \bar{x}_{k_i i})$, $i = 1, 2$ be the component of the sample mean vector $\bar{x}' = (\bar{x}_1', \bar{x}_2')$, we can easily show that the information statistic used to test the hypothesis $\mu' = \mu_0'$ is essentially

$$U(2) = n[(x_1 - \mu_{10})', (\bar{x}_2 - \mu_{20})'] \begin{pmatrix} (w_1 - z_1)I_1 + z_1E_1 & z_{12}E_{12} \\ z_{12}E_{12}' & (w_2 - z_2)I_2 + z_2E_2 \end{pmatrix}^{-1} \cdot \begin{bmatrix} (\bar{x}_1 - \mu_{10}) \\ (\bar{x}_2 - \mu_{20}) \end{bmatrix}, \tag{2.2}$$

where w_1, w_2 are average sample variances, and z_1, z_2, z_{12} are average sample covariances. After some algebra which does not present any particular difficulties, we get

$$\begin{aligned} U(2) = {} & \frac{n(\bar{x}_1 - \mu_{10})'(I_1 - k^{-1}E_1)(\bar{x}_1 - \mu_{10})}{w_1 - z_1} \\ & + \frac{n(\bar{x}_2 - \mu_{20})'(I_2 - k_2^{-1}E_2)(\bar{x}_2 - \mu_{20})}{w_2 - z_2} \\ & + n[(\bar{x}_1 - \mu_{10})'J_1, (\bar{x}_2 - \mu_{20})'J_2]S^{-1} \begin{bmatrix} J_1'(\bar{x}_1 - \mu_{10}) \\ J_2'(\bar{x}_2 - \mu_{20}) \end{bmatrix}, \end{aligned} \tag{2.3}$$

where J_i is a $1 \times k_i$ vector all of whose elements are unity for $i = 1, 2$; and

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \tag{2.4}$$

where

$$s_{ii} = k_i(w_i + (k_i - 1)z_i); \qquad (\bar{x}_i - \mu_{i0})'J_i = \sum_{s=1}^{k_i} (\bar{x}_{si} - \mu_{si0}); \qquad \text{for } i = 1, 2,$$

and $s_{12} = k_1k_2z_{12}$.

Further it can be shown for the structure $(u)_2$ that $U(2)$ is distributed like a linear sum of two independent F's plus Hotelling's T^2, or as the linear sum of three independent F's free of nuisance parameters in the null case.

$$U(2) = (k_1 - 1)F_{(k_1-1),(k_1-1)(n-1)} + (k_2 - 1)F_{(k_2-1),(k_2-1)(n-1)} + T^2 \quad (2.5)$$

or

$$U(2) = (k_1 - 1)F_{(k_1-1),(k_1-1)(n-1)} + (k_2 - 1)F_{(k_2-1),(k_2-1)(n-1)} + \frac{2(n-1)}{n-2} F_{2,n-2}. \quad (2.6)$$

$U(2)$ is asymptotically distributed like $\chi^2_{k_1+k_2}$ as $n \to \infty$. In the alternative case one or more of the F's are non-central F's.

The computation of $U(2)$ can be made from two analysis of variance tables of the type of Table 1 where each of the first two F's in (2.6) is $(n-1)(Q_1/Q_2)$. The T^2 statistic may be obtained by computing a mean variable across the first batch of k_1 variables and another across the second batch of k_2 variables for each vectorial observation. This gives us two variables with n observations from which we can compute Hotelling's T^2 in the usual manner.

The extension to equality of vector means of g populations, all having structure $(u)_2$, can be made without any difficulties and would involve a linear sum of independent F's plus the multivariate analysis of variance Hotelling's T_0^2 statistic (based on the trace of the product of the inverse of the within matrix and the between matrix), which tests the equality of means of g bivariate normal populations.

3. UNIFORM CASE OF ORDER $m(u)_m$

For completeness, we will indicate briefly the situation where the covariance structure is $(u)_m$, a partitioned symmetric matrix of matrices Λ_{ij} of order $k_i \times k_j$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1m} \\ \cdot & \cdot & & \vdots \\ & & \cdot & \vdots \\ & & & \Lambda_{mm} \end{bmatrix}, \quad (3.1)$$

where $\Lambda_{ii} = (\Phi_i - \beta_i)I_i + \beta_i E_i$ for $i = 1, \cdots m$, and $\Lambda_{ij} = \beta_{ij}E_{ij}$ for $i \neq j$. Testing the hypothesis that the vector mean of this multivariate normal population is some specified vector, the information criterion leads us to a statistic which is made up of a linear sum of F's plus a Hotelling T^2,

$$U(m) = \sum_{i=1}^{m} (k_i - 1)F_{k_i-1,(k_i-1)(n-1)} + T^2, \quad (3.2)$$

where

$$T^2 = \frac{m(n-1)}{n-m} F_{m,n-m}. \quad (3.3)$$

It is clear that when $k_i=1$ for $i=1, \cdots m$, $U(m)=T^2$. Further $U(m)$ is asymptotically χ_k^2 where $k=\sum k_i$ as $n\to\infty$.

Again the computation of $U(m)$ can be made from analysis of variance tables of the type of Table 1 to compute the first mF's, and T^2 can be computed from the m marginal means and the sample covariance structure among these variables.

The extension to g populations yields a statistic $U_g(m)$, which is composed of a linear sum of independent F's of the type of Table 2 plus an independent multivariate analysis of variance statistic T_0^2 which tests the equality of the g vector means each from a m-variate population.

4. APPLICATIONS

A. Some applications to mixed model analysis of variance situations will be indicated. Here it is of interest to test the null hypothesis that all of the components of the vector mean μ of a multivariate normal variable $x'=(x_1, x_2, \cdots, x_k)$ are equal. If we assume the following covariance structure

$$\Lambda = \begin{bmatrix} \sigma_1^2 & \sigma_1^2\rho_1 & \cdots & \sigma_1^2\rho_1 & \sigma_1\sigma_2\rho_2 \\ \vdots & \ddots & & \vdots & \vdots \\ & & \ddots & \sigma_1^2\rho_1 & \vdots \\ & & & \sigma_1^2 & \sigma_1\sigma_2\rho_2 \\ & & & & \sigma_2^2 \end{bmatrix} \tag{4.1}$$

i.e., $k-1$ variables have a uniform structure with the kth variable having a different variance but being equally correlated with the other $k-1$ variables. We may make the following transformation $x_k-x_j=y_j\, j=1, \cdots, k-1$. Hence

$$\left.\begin{aligned} V(y_j) &= \sigma_2^2 - 2\rho_2\sigma_1\sigma_2 + \sigma_1^2 \qquad \text{for } j = 1, \cdots, k-1 \\ \text{Cov. } (y_jy_i) &= \sigma_2^2 - 2\rho_2\sigma_1\sigma_2 + \rho_1\sigma_1^2 \qquad \text{for } i \neq j \\ E(y_j) &= \mu_k - \mu_j. \end{aligned}\right\} \tag{4.2}$$

Therefore the variables $y_1, \cdots, y_{k-1}$ are multivariate normal with uniform covariance structure $(u)_1$ and $Ey_j=0$, $j=1, \cdots, k-1$ if and only if $\mu_1=\cdots=\mu_k$. The test may thus be made on the null hypothesis that all of means of the y's are zero. This then is the case discussed in section 1, and the test statistic U is distributed as

$$U = (k-2)F_{k-2,(k-2)(n-1)} + F_{1,n-1}, \tag{4.3}$$

which is computed from an analysis of variance table of the y's, where all components of the assumed vector mean are zero.

B. Here we assume that the covariance structure is of a general serial correlation type

$$\Lambda = \sigma^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdot & \cdot & \cdot & \rho_{k-1} \\ & \cdot & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \rho_2 \\ & & & & & \cdot & \rho_1 \\ & & & & & & 1 \end{pmatrix}, \tag{4·4}$$

$$\left.\begin{aligned} \text{Cov. } (x_i x_j) &= \sigma^2 \rho_{|i-j|} && \text{for } i \neq j \\ V(x_i) &= \sigma^2 && i = 1, \cdots, k. \end{aligned}\right\} \tag{4.5}$$

Certain special cases are as follows:

1. Serial correlation case: $\rho_{|i-j|} = \rho^{|i-j|}$ or $\rho_i = \rho^i$.
2. Circular serial correlation case:

$$\rho_i = \frac{\rho^i + \rho^{k-i}}{1 + \rho^k} \qquad i = 1, \cdots, k-2, \ \rho_{k-1} = \rho_1.$$

See T. W. Anderson [1, p. 218].

3. Successive correlation case: $\rho_1 = \rho$ and $\rho_i = 0$, $i = 2, \cdots, k-1$. See Box [3] or Kamat and Sathe [5] for references to this case.

B_3. Where $k=3$ so that x_1, x_2, x_3 have means μ_1, μ_2, μ_3 we make the transformation $y_1 = x_2 - x_1$; $y_2 = x_3 - x_2$ noting that $\mu_2 - \mu_1 = \mu_3 - \mu_2 = 0$ if and only if $\mu_1 = \mu_2 = \mu_3$. Further

$$V(y_1) = V(y_2) = 2\sigma^2(1 - \rho_1), \tag{4.6}$$

hence we are in the case $(u)_1$ and the test statistic is the sum of two independent F's computed from an analysis of variance table on the y's,

$$U = F_{1,n-1} + F_{1,n-1}. \tag{4.7}$$

B_4. Where $k=4$ so that x_1, x_2, x_3, x_4 have means $\mu_1, \mu_2, \mu_3, \mu_4$, we make the transformation

$$y_1 = x_1 - x_3; \quad y_2 = x_2 - x_4; \quad y_3 = x_1 - x_2 + x_3 - x_4$$

noting that

$$\mu_1 - \mu_3 = \mu_2 - \mu_4 = \mu_1 - \mu_2 + \mu_3 - \mu_4 = 0$$

if and only if

$$\mu_1 = \mu_2 = \mu_3 = \mu_4.$$

5. REMARKS

It is to be noted that in all the analysis of variance type situations, one could have made a test for the hypothesis of equality of means even where the covariance matrix was arbitrary. The details of such a test are given by T. W. Anderson [1, p. 110]. For example, when $k=3$, for the same transformation used in this paper, one would get as a statistic $T^2=2(n-2)^{-1}(n-1)F_{2,n-2}$. Actually the two tests reflect the different methods that are appropriate in the bivariate case, for testing whether a vector mean is equal to some specified vector. Assuming that the two unknown variances are unequal, and the covariance is arbitrary, the test is Hotelling's T^2. However, on the supposition that the unknown variances are equal, the test statistic involves a pooled estimate for the common variance, and by section 1 turns out to be distributed as the sum of two independent F ratios.

The distribution of linear sums of independent F ratios, as well as other methods for specific serially dependent situations in the analysis of variance, are being currently investigated.

6. ACKNOWLEDGMENTS

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