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A Comprehensive Examination of Economic Harvest Optimization Simulation Methods

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ABSTRACT. The simulation algorithms that determine a harvest trajectory based on supply and demand—Economic Harvest Optimization (ECHO) and the present net worth (PNW) option of the Timber Resource Economic Estimation System—appear to offer reasonable approximations of optimal harvest regimes and to approach stable harvests. The two versions differ primarily in the calculation of growth on the marginal acre and in the way the algorithms iterate to a feasible solution. The impact of commercial thinning, shifting demand curves, and multiple sites and species is discussed relative to the capability of the algorithms to achieve optimality and stability. FOREST SCI. 27:523–536.

ADDITIONAL KEY WORDS. Present net worth, timber harvest scheduling, forest simulation, quadratic programming.

TIMBER HARVEST SCHEDULES which maximize discounted net stumpage returns when the harvest amount affects price in even-aged stands are currently defined by two simulation techniques. The two models, which approach the optimal solution through a binary search process, are ECHO (Economic Harvest Optimization) (Walker 1971, 1976) and the PNW (present net worth) option of TREES (Timber Resource Economic Estimation System) (Tedder, Schmidt, and Gourley 1979). This paper compares the two heuristics used to improve the algorithm performance: the restart method of ECHO and the quadratic interpolation method of the PNW option. It also explores conditions for arriving at optimal and stable harvest paths and offers some extensions.

THE HARVEST-SCHEDULING PROCESS

Harvest scheduling requires decisions on two aspects in each time period: the amount of harvest and the order of stands to be harvested. Both algorithms simplify the search for an optimal schedule by presuming that the harvest priority can be prespecified. The PNW option of TREES may specify harvest of oldest trees first, slowest growing trees first, or highest value trees first. In addition to the above options, the ECHO algorithm allows complete enumeration of harvest priority by a priori ordering for harvest of individual stands.

Once harvest priority is determined, the amount to be harvested in each period is found through a binary iterative search started by specifying an initial guess for the total quantity to be harvested in the first period. Stands are then harvested according to established priority until the harvest level for the period has been

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met. The last age class harvested in a period becomes a "linkage" age class, of which only part is harvested in the current period.

Leaving part of an age class for the next period is desirable only if the net discounted return equals what would be gained from harvest in the current period. That is, the change in value due to growth or different stumpage prices in the next period must offset the discount factor. To ensure this, the harvest in the second period is calculated using a linkage equation that takes into account growth rate, discount factor, and supply and demand interactions at the stumpage level.

Once the quantity of harvest is calculated for the second period, stands are again harvested until that quantity has been met and the linkage class to the third period is determined. The algorithm procedes in this manner, linking each harvest to the harvest of the previous period until the harvest called for is greater than the volume available or less than zero. For example, if in the first period the initial guess of harvest volume is higher than what will be cut in the final solution, the greater amount of acres cut and replanted will then be harvested again sooner than with the feasible solution. As they are harvested sooner, the disparity in growth rates creates a one-period volume growth ratio greater than the current discount rate. When this occurs, the harvest increases. The opposite holds when an initial guess is too low. (The mathematical conditions for this explanation will be presented later.) When the harvest called for is greater than the available volume, the algorithm begins again with the first-period harvest level adjusted downward; when the harvest called for is less than zero the first-period harvest level is adjusted upward. This process continues until initial harvest levels are found that bracket the optimum, one too high, the other too low. A trial harvest level between these levels is then selected and the harvest simulation begins again, ultimately to find whether the trial level is too high or low. The trial harvest level replaces the previous high/low level, cutting in half the difference between them, thus providing a binary search algorithm. This procedure continues until one of several stopping criteria is met.

ECHO and the PNW option in TREES differ somewhat in stopping criteria and in determination of the linkage class. In ECHO, iterations continue until the difference between high and low first-period harvest levels is narrowed to an arbitrarily designated minimum range. For the first period, that range is Δ (Fig. 1), and for periods 2 through N, the range is δ, where δ is usually greater than Δ. Because harvest trajectories only drift apart over time, not closer together, δ is used to promote computational efficiency. If d_i equals the differences between the high and low for any period i, then as long as $d_i \leq \delta$, the simulation continues. If $d_{i+1} > \delta$, then the ECHO restart capability takes over in period $i + 1$. In Figure 1, d_7 exceeds δ; therefore, the restart begins in period seven, starting with the inventory as it exists after period six. Harvest in period seven is not bound to that of period six. The iterative search and restart procedure continue until the schedule is determined for the desired number of periods.

In the PNW option of TREES, an interpolation rather than a restart procedure is used to vary growth rates of the linkage class in proportion to the amount harvested. This procedure tends to postpone divergence. Two stopping criteria are used: the search stops if a harvest schedule reaches the last period of the planning horizon without exhausting the inventory or becoming negative (Fig. 2A), or the schedule stops short of the planning horizon when the difference between possible first-period harvest levels becomes smaller than the specified limit, Δ (Fig. 2B). When the harvest schedule reaches the last period successfully, several conditions can be placed on the first stopping criterion. For example, one can specify that the harvest in the last period equal forest growth or that all

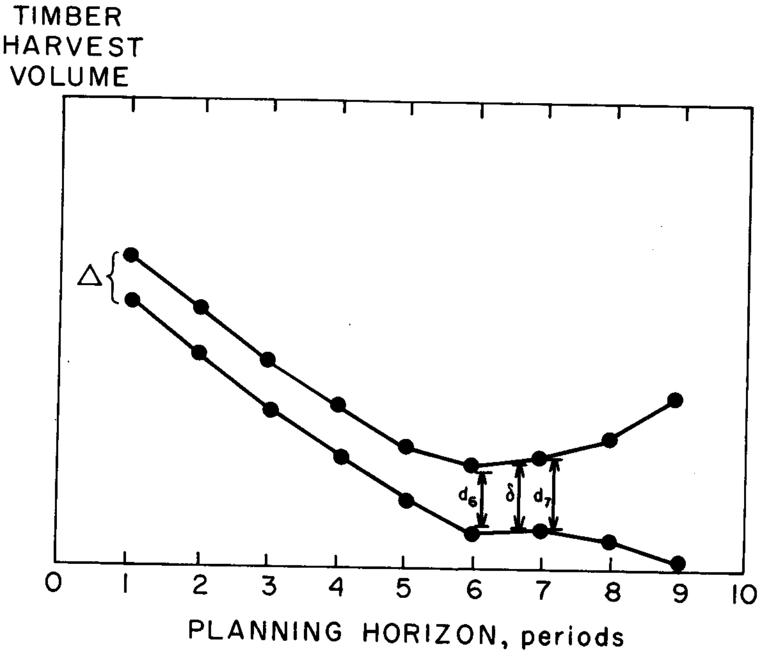

FIGURE 1. An illustration of the ECHO restart procedure. The difference between harvest schedules exceeds the minimum designated range (δ) in period 7. The optimal lower path is followed to period 6.

inventory above a given age class be exhausted. Specification of more restrictive ending conditions will cause the search to continue.

ECHO's restart procedure and the PNW growth-rate interpolation affect the capability of the algorithm to find optimal and stable harvest paths in different ways that we will discuss.

OPTIMIZING A HARVEST REGIME

Johnson and Scheurman (1977) have shown that the problem of optimizing a harvest schedule with a downward sloping linear demand curve can be formulated as a quadratic program. Because the optimality conditions for quadratic programming problems are well known, they were able to demonstrate the relationship between the simulation solution and the true optima.

Johnson and Scheurman's notation for the linkage equation, developed from the appropriate first-order condition, is

$$(C_1 - 2C_2h_j)V_{ij} - C_3V_{ij} - \lambda_i + l_i + \mu_{ij} = 0. \qquad \begin{array}{l} i = -M, \ldots, j-z \\ j = 1, \ldots, N. \end{array} \tag{1}$$

where C_1 = the price intercept of the demand curve. (For simplicity, assumed constant over time.)

C_2 = the slope of the demand curve. (For simplicity, assumed constant over time.)

h_j = total harvest in period j where $h_j = \sum_{i=-M}^{j-z} V_{ij}x_{ij}$.

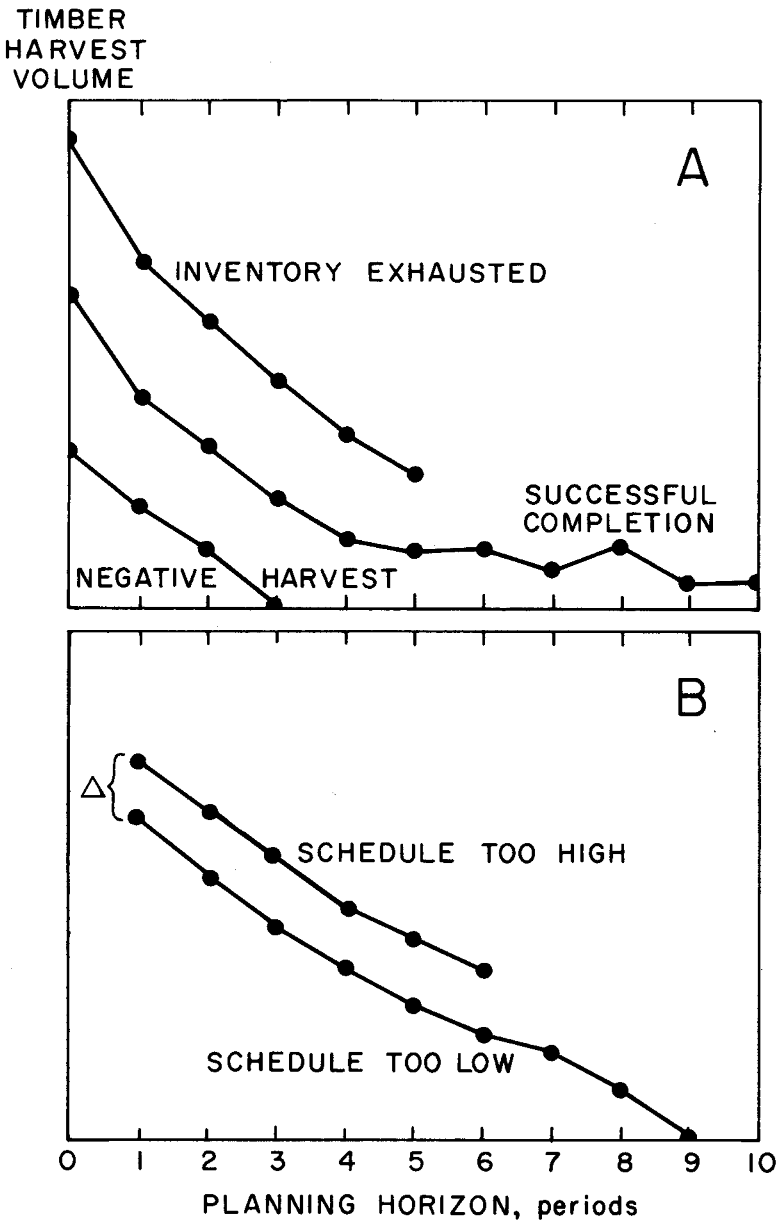

FIGURE 2. An illustration of the stopping criteria of the present-net-worth option of TREES. (A) A successful three-iteration search. (B) An unsuccessful search over the planning horizon. Search stops when the difference between the harvest levels is less than a specified limit (Δ).

V_{ij} = volume per acre, age class i, period j.

x_{ij} = acres cut in period j from age class i, the period of stand establishment that does not change as the stand ages.

C_3 = cost per unit of harvest. (For simplicity, assumed constant over time and with the age class harvested. Both the ECHO and PNW algorithms can include cost functions that are linear and nonlinear.

Nonlinear cost function could cause excessive problems in that the objective function could become a polynomial of higher degree than quadratic.)

λ_i = increase in the objective function due to a 1-acre increase in class i.

l_j = increase in the objective function from harvest in stands replacing class i due to a 1-acre increase in x_{ij}.

μ_j = total net discounted return (including current return and opportunity costs) from a 1-acre increase in class i, period j.

$-M$ = period of stand establishment for the oldest age class in the starting inventory. The minus indicates the age class was established before the start of the planning period.

N = number of periods in the planning horizon.

z = minimum number of periods before regenerated stands are eligible for harvest.

Equation (1) is equivalent to stating that discounted marginal net revenue must offset the opportunity costs of harvests.

The multipliers λ_i, l_j, μ_{ij} may be interpreted as the "shadow-prices" of the constraints. They represent the increase in the objective function achieved by relaxing the constraints one unit. An alternate view of λ_i is that it is the increase in net discounted return from future harvests due to a 1-acre decrease in age class i in period j. As such, λ_i is the opportunity cost of *cutting* 1 acre from age class i in period j. (The less cut from age class i in period j, the more cut in period $j + 1$.) The opportunity cost of *not cutting* an acre from age class i in period j is l_j. (The more cut now, the sooner regenerated stands will be available for cutting.)

The total undiscounted revenue (TR) in period j, assuming the present net worth approach, can be written

$$\mathrm{TR}_j = (C_1 - C_2 h_j) h_j.$$

The total undiscounted cost (TC) of harvest in period j can be written

$$\mathrm{TC}_j = C_3 h_j.$$

The discount factor is $D^j = (1 + I)^{P_j}$ where I is the annual discount rate and P is the period length. Noting that

$$\frac{\delta \mathrm{TR}}{\delta x_{ij}} = (C_1 - 2C_2 h_j) V_{ij},$$

we can define $(C_1 - 2C_2 h_j) V_{ij}$ as MR_{ij}, the marginal revenue in period j due to a 1-acre change in the acres harvested from age class i. Similarly,

$$\frac{\delta \mathrm{TC}}{\delta x_{ij}} = C_3 V_{ij}, \quad \text{or} \quad \mathrm{MC}_{ij},$$

the marginal change in total cost due to 1-acre change in the acres harvested from age class i in period j. Therefore, we can rewrite equation (1) (the last Kuhn-Tucker derivation from Johnson and Scheurman 1977)

$$\frac{\mathrm{MR}_{ij} - \mathrm{MC}_{ij}}{D^j} = \lambda_i - l_j - \mu_{ij} \qquad \begin{array}{l} i = -M, \ldots, j - z \\ j = 1, \ldots, N. \end{array} \tag{2}$$

That is, the marginal discounted net revenue from harvesting an additional acre of age class i in period j must equal the quantity on the right side of the equation.

Johnson and Scheurman (1977) give five general conditions which must be met for the ECHO and PNW algorithms to yield a harvest schedule equivalent to the quadratic-program solution. In addition, they discuss three that pertain to financial maturity, including land holding costs (Johnson and Scheurman 1977, p. 26):

1. Optimal preordering of harvests by stand class must be possible.
2. Every period must have a linkage class for which $x_{ij} > 0$ and $x_{ij+1} > 0$. (It is harvested in both the current and the succeeding period.)
3. For the linkage class, $(l_j - l_{j+1}) = 0$.

That is, the opportunity cost of delaying the regeneration and future harvests on an acre of the linkage age class is constant between periods j and $j + 1$. Since constant costs do not affect optimality, assuming that $l_j = l_{j+1}$ is equivalent to ignoring the opportunity cost of delaying future harvests. Condition 3 is therefore equivalent to "simple financial maturity," the type A analysis in Duerr (1960). Suppose we know the harvest level in period j. Then, with condition 1, we can determine the last age class harvested in period j, x_{ij}. Age class i constitutes the linkage age class.

From equation (2) we can derive a similar condition for x_{ij+1}

$$\frac{\mathrm{MR}_{ij+1} - \mathrm{MC}_{ij+1}}{D^{j+1}} - \lambda_i + l_{j+1} + \mu_{ij+1} = 0. \tag{3}$$

From (b) of the Kuhn-Tucker conditions (see Johnson and Scheurman 1977) we derive

$$\mu_{ij}x_{ij} = 0 \quad \text{and} \quad \mu_{ij+1}x_{ij+1} = 0.$$

With condition 2, $x_{ij} > 0$ and $x_{ij+1} > 0$. Therefore, μ_{ij} and $\mu_{ij+1} = 0$. Equating (2) and (3), we can write

$$\frac{\mathrm{MR}_{ij} - \mathrm{MC}_{ij}}{D^j} = \frac{\mathrm{MR}_{ij+1} - \mathrm{MC}_{ij+1}}{D^{j+1}} - (l_j - l_{j+1}). \tag{4}$$

With condition 3, $(l_j - l_{j+1}) = 0$. We can rewrite (4) as

$$\frac{\mathrm{MR}_{ij} - \mathrm{MC}_{ij}}{D^j} = \frac{\mathrm{MR}_{ij+1} - \mathrm{MC}_{ij+1}}{D^{j+1}}. \tag{5}$$

We can then substitute terms from equation (1) in equation (5):

$$\frac{(C_1 - 2C_2h_j)V_{ij} - C_3V_{ij}}{D^j} = \frac{(C_1 - 2C_2h_{j+1})V_{ij+1} - C_3V_{ij+1}}{D^{j+1}}. \tag{6}$$

This equation may be interpreted as requiring that an acre cut from age class i return the same discounted marginal net revenue in each harvest period.

Because we have already chosen h_j and because C_1, C_2, C_3, V_{ij}, V_{ij+1}, D^j, and D^{j+1} are known, we can solve for the required harvest level in $j + 1$ to satisfy the linkage equation. Let $C_4 = C_1 - C_3$.

$$h_{j+1} = \frac{C_4}{2C_2}\left[1 - D\left(\frac{V_{ij}}{V_{ij+1}}\right)\right] + D\left(\frac{V_{ij}}{V_{ij+1}}\right)h_j. \tag{7}$$

We need to examine the extent to which the three conditions hold. If they are not met, how close do the ECHO and PNW algorithms come to finding an optimal harvest sequence?

Condition 1.—Walker (1971) states that if volume per acre increases with age at a decreasing rate, a simple oldest-first harvest priority will be optimal. For the simple case we are considering, this is equivalent to harvesting age classes in order of decreasing marginal value growth percent (MVGP) and is optimal only if we accept the "simple financial maturity" criterion. Furthermore, if we introduce variable harvest costs and quality premiums (an increase in value per unit of wood derived from an increase in wood quality as trees mature) or if multiple site, species types, stocking levels, and management regimes are included, the relationship between MVGP and age may not be straightforward. MVGP may not decrease with age in a given site-type and may vary for a given age class in different site types. While the algorithms offer flexibility in specifying harvest priority, neither offers the option of ordering strictly on the basis of MVGP. Even should that option be offered, it would ignore the growth of stands replacing the current stands. Because of the faster growth of replacement stands, a truly optimal ordering would likely give higher priority to cutting high-site stands than to cutting those with decreasing MVGP.

Condition 2.—This condition is central to the operation of the algorithm. If violated, the linkage between period harvest levels is broken. For the ECHO and PNW algorithms, this occurs when no harvest schedule can be found that satisfies the linkage equation in all periods. The PNW algorithm terminates search on the implicit assumption that if the condition is not met, no criterion exists for choosing optimal harvest levels. ECHO, in contrast, makes use of the restart procedure because the algorithm exactly exhausts the last age-class cut at the point of divergence, which is the logical place to terminate the linkage relationship and to start the search anew. The harvest path to the point of divergence may not be optimal but is as close as possible within the limits of accuracy inherent in the simulation process.

ECHO's restart method appears to have considerable merit. As we shall see more clearly in the discussion of achieving a stable harvest level, divergence between harvest sequences can be caused by differences in the last class harvested. Harvest schedules will remain parallel as long as the last class harvested in each period is the same. However, the schedule with a higher first-period harvest will cut more of the last class than a schedule with a lower first-period harvest. Eventually, the higher schedule will cut into an age class beyond the lower schedule, and at that point, differences in the growth rates of the linkage class will cause divergence.

ECHO's restart method has the practical advantage of always producing a feasible harvest schedule. Once the over-age inventory has been reduced, however, restart may be required every period. In such circumstances, a simple harvest schedule (such as harvest of all stands whose MVGP is less than the discount rate) may prove more satisfactory.

Condition 3.—Assuming that delay of future harvests has no opportunity, cost will, in general, cause harvests to be lower than optimal in the near term and higher than optimal later. In general, the lower the discount rate and the higher the growth rate of a forest, the greater will be the difference between the optimal harvest schedule and the simple financial maturity approach.

A Fourth Condition.—To the three conditions we have discussed, a fourth must be added to cover the PNW algorithm that operates over a fixed time horizon. If a feasible harvest schedule can be found that satisfies the linkage equation for the planning horizon, the algorithm stops. The assumption is that the first feasible solution is optimal. It is easy to demonstrate that this assumption is not valid for short-time horizons. Many feasible harvest schedules can be found by simply changing the starting harvest level. To remedy this, the planning horizon must be lengthened, which makes satisfying the linkage equation in all periods more difficult and which forces the algorithm to iterate to the present net worth maximizing solution.

Stabilizing a Harvest Regime

Traditionally, forest managers have had the objective of a "regulated" forest—one with a uniform distribution of age classes allowing stable harvest levels indefinitely. Walker (1971) states that ECHO "defines the optimal time path to a long-run equilibrium forest." In this section, we will examine the conditions under which operation of the algorithm results in a regulated forest supporting stable periodic harvest.

Sessions (1977) outlines conditions for the linkage equation (7) to produce even-flow harvest levels.

$$\text{Let } A = \frac{D}{(V_{ij+1}/V_{ij})} = \frac{\text{the one-period discount factor}}{\text{the one-period volume-growth ratio}}.$$

We can restate his conditions for a stable harvest level when marginal net revenue is positive.

(1) A stand class exists for which $A = 1$ (the equilibrium class).
(2) The equilibrium class must be the last class harvested in a period j.
(3) The harvest level in period j must be in balance with the volume growth of the forest.
(4) The forest must be sufficiently regulated by period j so that growth in classes cut before the equilibrium class (the last cut in all subsequent periods) varies less than the volume of the equilibrium class in subsequent periods $j, j + 1, \ldots, N$.

The simultaneous fulfillment of these conditions is not guaranteed by satisfying the linkage relationship. First, condition 1 may not hold if the discount rate is high. No age classes may be growing fast enough so that $A = 1$. More commonly, because of discrete age classes, there may be some for which $A > 1$ and $A < 1$, but none for which $A = 1$.

The use of annual age classes and harvest periods minimizes discontinuity between age-class growth rates and makes it more likely an age-class growth rate will approach the discount rate. A second strategy, adjusting the discount rate to equal the growth rate of some age classes, accomplishes the same thing. A third strategy in the PNW algorithm is to make age classes continuous for calculating growth rates. It is assumed that the growth rate of the last acre cut depends on the proportion of the last age class harvested. A quadratic function is used to estimate V_{ij+1}/V_{ij} for the last acre cut (Fig. 3). If exactly half of age class i were cut in period j, the growth ratio would be unadjusted: V_{ij}/V_{ij+1}.

While all strategies mentioned increase the stability of the algorithm, none achieves it completely. Even with annual classes, there may be none for which $A = 1$. Adjusting the discount rate or interpolating growth rates may ensure that $A = 1$ for a class, but adjusting the discount rate introduces error into the discounting procedure. Interpolation ensures absolute stability only if the last age class harvested is in correct proportion. This may be impossible to achieve because small differences in the initial level can result in large differences 10 or 20 periods later.

Even if conditions 1 and 2 for stability hold, conditions 3 and 4 may not. If condition 4 is met, but not condition 3, the harvest level in period j does not equal the volume growth of the forest. A harvest level in excess of growth means

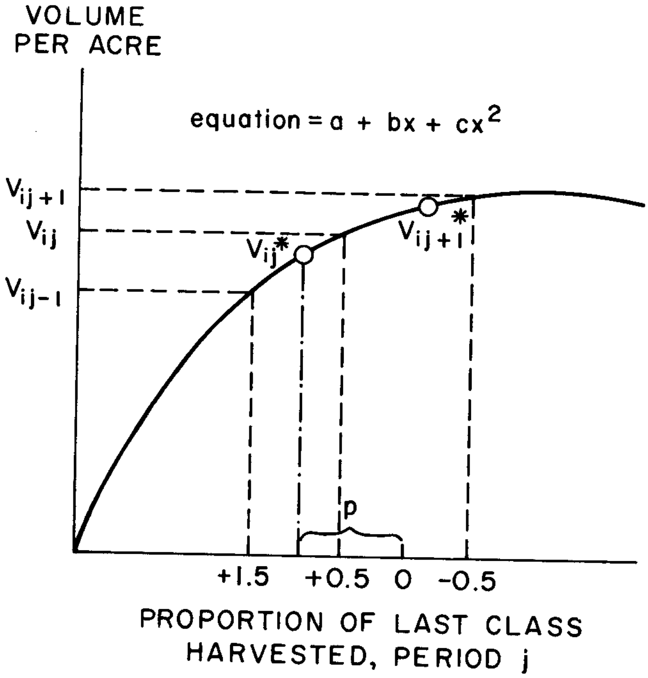

FIGURE 3. Interpolation routine of the present-net-worth option. A quadratic curve is fit to three points: $(1.5V_{ij-1})$, $(0.5V_{ij})$, $(-0.5V_{ij+1})$. If p is the proportion of the last class harvested, the growth ratio is V_{ij+1}^*/V_{ij}^*.

the age of the last class harvested will fall below equilibrium age ($A < 1$). Harvest levels will rise, further deplete the inventory, further increase A, and eventually will exhaust the inventory. A harvest level less than growth will mean the age of the last class harvested will rise above equilibrium age ($A > 1$). Harvest levels will fall, further increase the inventory, further reduce A, and eventually will reach negative levels.

If condition 3 is met, but not 4, harvests may oscillate as growth in the classes cut before equilibrium age rises and falls. Because growth above harvest levels means declining harvest, and growth below harvest levels means increasing harvests, the oscillations tend to widen in amplitude. As the planning horizon lengthens, the harvest level will eventually exceed available inventory or become negative.

ECHO's restart procedure may allow a pseudo-equilibrium to be maintained indefinitely, even when no class exists for which $A = 1$. Suppose, for class i, $A > 1$ and for class $i - 1$, $A < 1$. Divergence between harvest schedules tends to occur at the point class i is exhausted. ($A > 1$ signals a decrease in harvest; $A < 1$ signals an increase.) Restarting the algorithm after class i is exhausted will tend to produce the same result in the next period—divergence will occur when class $i-1$ is exhausted. Repetition of the search and restart process will result in harvest of the last class for which $A > 1$ in each period. If the forest is approximately regulated when this process begins, harvests will be regulated as well. If not, irregularities in the forest structure will be echoed in the harvest amounts.

Although neither the PNW's quadratic interpolation nor ECHO's restart can assure perfect regulation of the forest, the harvest paths resulting from the search process do tend toward regulation. If the forest is initially overaged, $A > 1$, harvest levels decrease over time. Because volume per acre usually decreases as

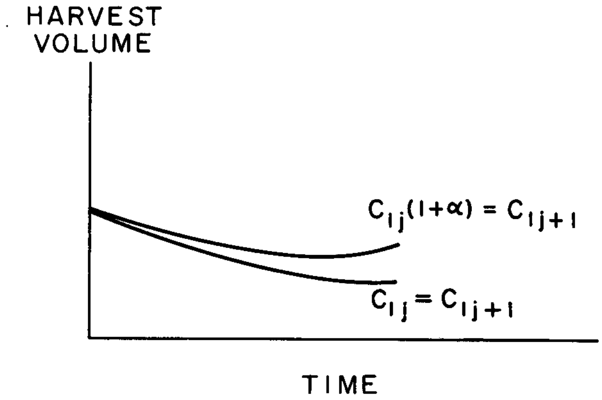

FIGURE 4. The effect of shifts in the demand intercept when the starting level is held constant.

younger age classes are cut, the number of acres cut each year falls less rapidly. As the equilibrium class is approached and A nears 1, changes in harvest levels become less and the forest approaches regulation.

ALGORITHM PERFORMANCE IN COMPLEX SITUATIONS

We have been dealing with a simple situation: stable demand, no thinning, price or costs unvarying with stand age, and one site class and species type. Introduction of more realistic elements requires modification of our conclusions on optimality and stability based on the simple model.

Shifting Demand.—In both ECHO and the PNW option of TREES, the demand intercept may be changed over time. Introducing a demand shifter does not affect the capability of the algorithm to optimize harvest levels but merely adds a new term in the linkage equation. It does, however, alter the harvest path and the conditions for stable harvest levels.

Let α be the rate of a change in the demand equation intercept, so that $(1 + \alpha)\, C_{1j} = C_{1j+1}$. Substituting into equation (6), we derive the new linkage equation:

$$\frac{(C_{1j} - 2C_2h_j)V_{ij} - C_3V_{ij}}{D^j} = \frac{[(1 + \alpha)C_{1j} - 2C_2h_{j+1}]V_{ij+1} - C_3V_{ij+1}}{D^{j+1}} \tag{8}$$

and, letting $C_4 = (C_{1j} - C_3)$,

$$h_{j+1} = \frac{C_4}{2C_2}\left(1 - D\frac{V_{ij}}{V_{ij+1}}\right) + D\frac{V_{ij}}{V_{ij+1}}h_j + \frac{\alpha C_{1j}}{2C_2}. \tag{9}$$

Thus, h_{j+1} will be increased by $\frac{\alpha C_{1j}}{2C_2}$ when α is positive and the demand intercept shifts by ($\alpha \times 100$) percent (Fig. 4).

The starting harvest level may need to be reduced to maintain feasible harvest levels. An upward shift in the demand curve intercept also raises the age of the equilibrium age class.

Letting $A = \dfrac{D}{V_{ij+1}/V_{ij}}$,

$$h_{j+1} = \frac{C_4}{2C_2}\left(1 - A\right) + Ah_j + \frac{\alpha C_{1j}}{2C_2}. \tag{10}$$

Rearranging the terms in (8) and noting that $Ah_j = (1 - A)(-h_j) + h_j$,

$$h_{j+1} = h_j + (1 - A)\left(\frac{C_4}{2C_2} - h_j\right) + \frac{\alpha C_{1j}}{2C_2}. \tag{11}$$

Therefore, h_{j+1} will equal h_j when

$$(1 - A)\left(\frac{C_4}{2C_2} - h_j\right) + \frac{\alpha C_{1j}}{2C_2} = 0,$$

or equivalently, when

$$A = 1 + \frac{\alpha C_{1j}}{C_4 - 2C_2 h_j}.$$

Assuming that the marginal net revenue is positive ($C_4 - 2C_2h_j > 0$) and that $\alpha > 0$, then $A > 0$ when $h_{j+1} = h_j$. The growth rate must be less than the discount rate.

One consequence of a demand curve shifting over time, therefore, is that harvest must be less than growth for stable harvest levels to be maintained. A forest with equal acres in each age class will not reach stable harvest levels.

Age-Dependent Stumpage Price and Harvest Costs.—Introduction of age dependent prices and harvest costs has the same effect as a shift in demand. The effect of a quality premium which increases the stumpage price per unit as the age (size) of timber increases is shown in Figure 5. Both algorithms use the same approach.

Let BP_{ij} = base per unit of volume harvested, age class *i*, period *j* (the price received in the base year for timber of the same age as age class *i* in year *j*).
$\mathrm{BP}_{\mathrm{av}}$ = price per unit of volume received for the average age tree in the base year.
CA_{ij} = cost per acre harvested from age class *i*, period *j*. (Assume that cost per acre for a given age remains constant over time, but cost per acre may change with stand age.)
CU_{ij} = cost per unit of volume harvested, age class *i*, period *j*. (Assume that cost per unit of volume for a given age remains constant over time, but cost per unit may change with stand age.)
MC_{ij} = the change in total cost in period *j* due to a 1-acre change in the harvest from age class *i*.

$$\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{\mathrm{av}}} \cdot \frac{(C_{1j} - 2C_2h_j)V_{ij}}{D^j} - \frac{\mathrm{CA}_{ij} + \mathrm{CU}_{ij}V_{ij}}{D^j}$$
$$= \frac{\mathrm{BP}_{ij+1}}{\mathrm{BP}_{\mathrm{av}}} \cdot \frac{(C_{1j} - 2C_2h_{j+1})V_{ij+1}}{D^{j+1}} - \frac{\mathrm{CA}_{ij+1} + \mathrm{CU}_{ij+1}V_{ij+1}}{D^{j+1}}. \tag{11}$$

Let $\mathrm{MC}_{ij} = \mathrm{CA}_{ij} + \mathrm{CU}_{ij}V_{ij}$,

$$\mathrm{MC}_{ij+1} = \mathrm{CA}_{ij+1} + \mathrm{CU}_{ij+1}V_{ij+1}.$$

Rearranging terms, we have

$$h_{j+1} = \frac{C_{1j}}{2C_2}\left[1 - D\left(\frac{V_{ij}}{V_{ij+1}}\right)\left(\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{ij+1}}\right)\right] + D\left(\frac{V_{ij}}{V_{ij+1}}\right)\left(\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{ij+1}}\right)h_j$$
$$+ \left(\frac{\mathrm{BP}_{\mathrm{av}}}{\mathrm{BP}_{ij+1}}\right)\left(\frac{1}{2C_2}\right)\left(\frac{1}{V_{ij+1}}\right)\left[D(\mathrm{MC}_{ij}) - \mathrm{MC}_{ij+1}\right]. \tag{12}$$

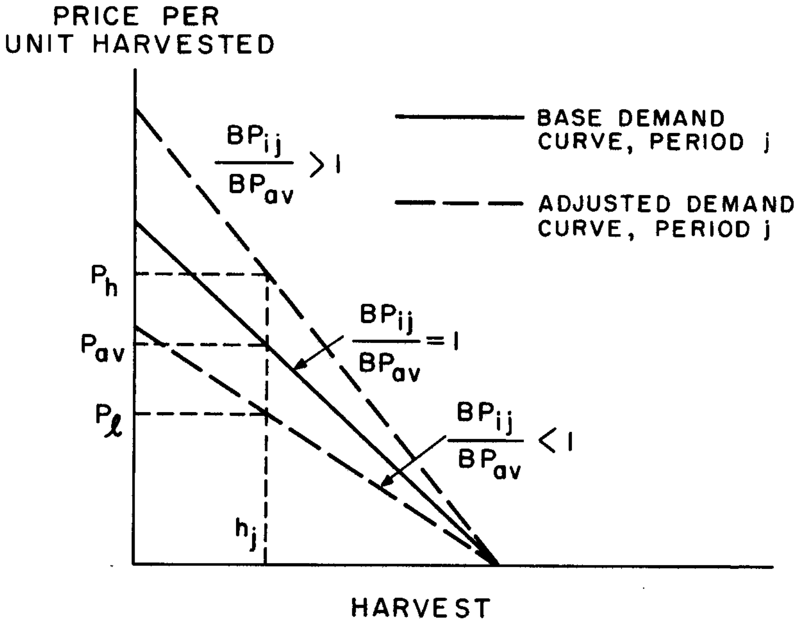

FIGURE 5. Price per unit harvested adjusted to reflect the quality premium in the base year.

Let $R = \dfrac{D}{(V_{ij+1}/V_{ij})(\mathrm{BP}_{ij+1}/\mathrm{BP}_{ij})} = \dfrac{\text{discount factor}}{\text{value growth ratio}}.$

Then

$$h_{j+1} = \frac{C_{1j}}{2C_2}(1 - R) + Rh_j = \frac{\mathrm{BP}_{av}}{\mathrm{BP}_{ij+1}} + \frac{D(\mathrm{MC}_{ij}) - \mathrm{MC}_{ij+1}}{2C_2V_{ij+1}}. \tag{13}$$

In period *j*, when the last class harvested is such that $\mathrm{BP}_{ij} = \mathrm{BP}_{av}$, the base demand curve is used for determining the stumpage value received. If $\mathrm{BP}_{ij} > \mathrm{BP}_{av}$, the slope of the demand curve is increased. If $\mathrm{BP}_{ij} < \mathrm{BP}_{av}$, the slope is decreased to calculate harvest in the next period.

Use of quality premiums and age-dependent costs does not affect the optimality of the algorithm, although harvest schedules will change. In general, the quality premium has an effect similar to increasing the growth rate of all classes. Introduction of harvest costs that decrease per unit of volume as stand age increases has an effect similar to shifting the demand curve away from the origin. For a given h_j, h_{j+1} will be larger to compensate for the increase in net value. Initial harvest levels will be adjusted downward to preserve feasibility.

Use of quality premiums and age-dependent costs does alter the conditions for stability of the linkage equation. Assume $R = 1$. Substituting into equation (13), we have

$$h_{j+1} = h_j + \frac{\mathrm{BP}_{av}}{\mathrm{BP}_{ij+1}} \frac{D(\mathrm{MC}_{ij}) - \mathrm{MC}_{ij+1}}{2C_2V_{ij+1}}.$$

Therefore, h_{j+1} will equal h_j if $D(\mathrm{MC}_{ij}) - \mathrm{MC}_{ij+1} = 0$. That is,

$$\frac{D/\mathrm{MC}_{ij+1}}{\mathrm{MC}_{ij}} = 1.$$

Rearranging the terms in (8) and noting that $Ah_j = (1 - A)(-h_j) + h_j$,

$$h_{j+1} = h_j + (1 - A)\left(\frac{C_4}{2C_2} - h_j\right) + \frac{\alpha C_{1j}}{2C_2}. \tag{11}$$

Therefore, h_{j+1} will equal h_j when

$$(1 - A)\left(\frac{C_4}{2C_2} - h_j\right) + \frac{\alpha C_{1j}}{2C_2} = 0,$$

or equivalently, when

$$A = 1 + \frac{\alpha C_{1j}}{C_4 - 2C_2 h_j}.$$

Assuming that the marginal net revenue is positive ($C_4 - 2C_2h_j > 0$) and that $\alpha > 0$, then $A > 0$ when $h_{j+1} = h_j$. The growth rate must be less than the discount rate.

One consequence of a demand curve shifting over time, therefore, is that harvest must be less than growth for stable harvest levels to be maintained. A forest with equal acres in each age class will not reach stable harvest levels.

Age-Dependent Stumpage Price and Harvest Costs.—Introduction of age dependent prices and harvest costs has the same effect as a shift in demand. The effect of a quality premium which increases the stumpage price per unit as the age (size) of timber increases is shown in Figure 5. Both algorithms use the same approach.

Let BP_{ij} = base per unit of volume harvested, age class *i*, period *j* (the price received in the base year for timber of the same age as age class *i* in year *j*).

$\mathrm{BP}_{\mathrm{av}}$ = price per unit of volume received for the average age tree in the base year.

CA_{ij} = cost per acre harvested from age class *i*, period *j*. (Assume that cost per acre for a given age remains constant over time, but cost per acre may change with stand age.)

CU_{ij} = cost per unit of volume harvested, age class *i*, period *j*. (Assume that cost per unit of volume for a given age remains constant over time, but cost per unit may change with stand age.)

MC_{ij} = the change in total cost in period *j* due to a 1-acre change in the harvest from age class *i*.

$$\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{\mathrm{av}}}\cdot\frac{(C_{1j} - 2C_2h_j)V_{ij}}{D^j} - \frac{\mathrm{CA}_{ij} + \mathrm{CU}_{ij}V_{ij}}{D^j}$$

$$= \frac{\mathrm{BP}_{ij+1}}{\mathrm{BP}_{\mathrm{av}}}\cdot\frac{(C_{1j} - 2C_2h_{j+1})V_{ij+1}}{D^{j+1}} - \frac{\mathrm{CA}_{ij+1} + \mathrm{CU}_{ij+1}V_{ij+1}}{D^{j+1}}. \tag{11}$$

Let $\mathrm{MC}_{ij} = \mathrm{CA}_{ij} + \mathrm{CU}_{ij}V_{ij}$,

$$\mathrm{MC}_{ij+1} = \mathrm{CA}_{ij+1} + \mathrm{CU}_{ij+1}V_{ij+1}.$$

Rearranging terms, we have

$$h_{j+1} = \frac{C_{1j}}{2C_2}\left[1 - D\left(\frac{V_{ij}}{V_{ij+1}}\right)\left(\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{ij+1}}\right)\right] + D\left(\frac{V_{ij}}{V_{ij+1}}\right)\left(\frac{\mathrm{BP}_{ij}}{\mathrm{BP}_{ij+1}}\right)h_j$$
$$+ \left(\frac{\mathrm{BP}_{\mathrm{av}}}{\mathrm{BP}_{ij+1}}\right)\left(\frac{1}{2C_2}\right)\left(\frac{1}{V_{ij+1}}\right)\left[D(\mathrm{MC}_{ij}) - \mathrm{MC}_{ij+1}\right]. \tag{12}$$

The increase in marginal cost of harvesting the last acre must be equal to the one-period discount factor. In practice, this condition is unlikely to be met. If $R = 1$, and $\mathrm{BP}_{ij+1} > \mathrm{BP}_{ij}$, then $V_{ij+1}/V_{ij} < D$ (that is, volume growth is less than the discount factor). If cost per unit of volume harvested is presumed to decrease as volume per acre increases, costs per acre must be rising more slowly than volume per acre. In most cases when $R = 1$, $\mathrm{MC}_{ij+1}/\mathrm{MC}_{ij} < D$, and $h_{j+1} > h_j$. To maintain stability, the equilibrium age class must be such that the value growth is less than the discount factor $R < 1$. How much R must be increased depends on the relationship between value growth and harvest cost per acre. If $R > 1$, $\mathrm{MC}_{ij+1}/\mathrm{MC}_{ij}$ will likely be less than D, making the third term of equation (11) positive. However, the first term will usually be negative and larger than the third term.

Commercial Thinning.—Commercial thinning is included in the general algorithm by setting the thinning specification before beginning the search. The thinning volume must be harvested in each period (the "exogenous" harvest). If the linkage equation calls for an amount greater than the thinning specified, the thinning volume is removed and the remaining volume is harvested according to priority. If the linkage equation calls for an amount less than the thinning volume, the search procedure begins again with a higher initial harvest.

Including commercial thinning as exogenous harvest can be optimal, of course, only if the proposed thinning regime is optimal. Multiple runs with different thinning rules may be required to approximate an optimal solution. Stand-level analyses of optimal thinning regimes are not sufficient to determine the regime for an optimal harvest schedule. Thinnings may displace harvests of higher valued material and lead to less than optimal solutions.

Generally, exogenous harvests introduce a random element that may make forest regulation difficult or impossible because of increased instability.

Multiple Sites and Species.—Multiple sites and species makes harvest preordering more difficult. Existing algorithms tend to employ old rules-of-thumb, such as oldest first, slowest growing first, or highest value first, that offer reasonable approximations to optimal ordering. The PNW option of TREES differs significantly from ECHO in treatment of individual stands. Volumes on all harvest units are summed together by age class and species type.

The age class and species type with which harvest cutting ends becomes the linkage age class and species type. The proportion harvested from the linkage age class of the summed management units is applied to each age and species type over all management units (each unit is classified by site, species type, ownership, location, and type of administration). Resulting harvest volumes, harvest costs, and base revenues are adjusted by quadratic interpolation of the standard values for the appropriate management intensities. The adjusted quantities are summed over the entire harvest unit. The sums of harvest volumes in the last class, harvest costs, base harvest costs, and base price total revenues are then divided by the acres harvested in the final age class for values per acre. The final values are weighted averages for individual management units.

Averaging stands together allows many to be treated efficiently, but though stability is easier to obtain, the weighted average means a true optimum will not be reached. The cost of efficiency is loss of appropriate harvest priorities for individual site types. For example, the "slowest growing first" priority can be applied only by age class and species type—not by site class, stocking level, or management intensity. Consequently, solutions are likely to be less than optimal with this approach.

CONCLUSION

When downward sloping linear demand for stumpage is appropriate, the ECHO and PNW algorithms appear to offer reasonable approximations of the optimal harvest regime based on a simple financial maturity criterion. The optimal solution requires that several conditions be met. With complex mixtures of stands, the optimal harvest priority may be difficult to specify. Over long time periods, harvest schedules for which the linkage equation is satisfied in every period are difficult to find. Under such circumstances, the PNW algorithm does not find a feasible harvest schedule for the entire planning horizon, while ECHO's restart procedure does. But linkage equation criteria do not include the impact of future rotations. The result is likely to be a harvest schedule lower than optimal in early periods and higher than optimal later. The overall impact, however, may be minor. The possibility for multiple solutions in the PNW algorithm can be expanded by lengthening the planning horizon. Ironically, this may result in a harvest schedule that is closer to the optimum but that does not span the entire planning horizon.

The general approach has significant advantages over the quadratic programming approach in the ability to handle large problems (such as many stand classes) with relative ease. Costs are generally lower than for equivalent quadratic programming because no matrix inversion is required. The two versions differ primarily in calculation of growth on the marginal acre and in behavior once the algorithm iterates to an infeasible solution.

Neither method guarantees forest regulation or stable harvests, but both achieve these with reasonable success in many instances. The treatment of commercial thinning as an exogenous harvest makes optimizing and stabilizing harvest schedules difficult if not impossible. However, introduction of shifting demand curves, quality premiums, and age-dependent harvest costs does not cause any deterioration in performance.

LITERATURE CITED

DUERR, W. A. 1960. Fundamentals of forestry economics. McGraw-Hill, New York. 579 p.

JOHNSON, K. N., and H. L. SCHEURMAN. 1977. Techniques for prescribing optimal timber harvest and investment under objectives—discussion and synthesis. Forest Sci Monogr 18, 31 p.

SESSIONS, J. 1977. Stability considerations in ECHO modeling. Forest Sci 23:446–449.

TEDDER, P. L., J. S. SCHMIDT, and J. GOURLEY. 1979. TREES: timber resource economic estimation system. Vol I. A user's manual for forest management and harvest scheduling. Oreg State Univ, Corvallis, For Res Lab Bull 31a, 81 p.

WALKER, J. L. 1971. An economic model for optimizing the rate of timber harvesting. Ph D thesis, Univ Wash, Seattle. 117 p. Diss Abstr 32(5):2276-A (Microfilm 71-28489).

WALKER, J. L. 1976. ECHO: solution technique for a nonlinear economic harvest optimization model. *In* Systems analysis and forest resource management (J. Meadows, B. Bare, K. Ware, and C. Row, eds), p 172–188. Soc Am For, Wash, D.C. 457 p.